

Pitfalls of the Relaxation Time Approximation: Hydrodynamic Sound in a Multicomponent Fermi Liquid

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When transport in a Fermi liquid is treated in the relaxation time approximation, the quasiparticle energy appearing in the local equilibrium distribution must have the form determined by the nonequilibrium distribution function. Sometimes this requirement is overlooked and the equilibrium quasiparticle energy is used. In applications to unpolarized normal ^3He the resulting error can be repaired by a simple rescaling of the relaxation rates $1/\tau_l$ by the Fermi liquid corrections $1 + F^l/(2l+1)$. The distinction between the two forms of the relaxation time approximation is thus of little consequence, and quantities independent of the relaxation time are entirely unaffected. We point out that more significant damage results from using this wrong relaxation time approximation in a multicomponent (or spin-polarized single-component) Fermi liquid. In particular, it is essential to use the correct form to derive the velocity of hydrodynamic sound, even though the incorrect form also satisfies all the conservation laws, and even though the sound velocity is independent of the relaxation time.

1. INTRODUCTION

In treating nonequilibrium phenomena in a Fermi liquid with Landau's kinetic equation, it is sometimes convenient to replace the full collision integral $I(n)$ by a simple relaxation time approximation. Dating all the way back to the review of Abrikosov and Khalatnikov,¹ it has often been the practice to make this relaxation time approximation in the form

$$I(n) = -(1/\tau)[\delta n - \langle \delta n \rangle - 3\hat{p} \cdot \langle \hat{p} \delta n \rangle] \quad (1)$$

Here δn is the deviation of the quasiparticle distribution function n from uniform equilibrium, and the angular brackets denote an average over all directions of \hat{p} .

The justification for this form is that it represents collisions in a way that conserves number, momentum, and energy. There is, however, a second essential feature of the full collision integral that the approximation (1) fails to preserve, namely that $I(n)$ should vanish whenever n has the local equilibrium form. This second condition can be achieved by a suitable redefinition of δn in (1), as is done in some of the more recent treatments of Fermi liquid theory.^{2,3} The fact that the incorrect form of $I(n)$ in Eq. (1) is used in some of the classic papers in the field, however, has not always been emphasized.

The reason for the lack of attention to this point is probably this: if the relaxation time approximation is generalized to one in which each component in the spherical harmonic expansion of δn on the Fermi sphere relaxes with its own l -dependent relaxation time, then the incorrect form (1) can be converted to the correct relaxation time approximation by dividing each τ_l by the Fermi liquid correction $1 + F^l/(2l+1)$. Since the τ_l are only phenomenological parameters to begin with, this redefinition generally has no significant consequences.

The situation is quite different, however, when one deals with a multicomponent Fermi liquid, and it is our purpose in this note to call attention to this fact. If the Fermi liquid consists of two or more independently conserved components, then the error introduced by making the relaxation time approximation in the form of the many-component analog of Eq. (1) will in general lead to an incorrect result for a quantity—the velocity of hydrodynamic sound—in which the relaxation time does not appear at all. Such an error cannot, of course, be repaired by using a redefined relaxation time. To derive the correct velocity of sound in the multicomponent system it is essential to use a relaxation time approximation that has built into it not only the conservation laws, but also the fact that collisions maintain local equilibrium.

It seems timely to make this cautionary point, since there has been interest in recent years in the two-component Fermi liquid consisting of strongly polarized ^3He .⁴ Our hydrodynamic example applies to such a system at frequencies large compared with the spin relaxation rate.

2. HYDRODYNAMIC SOUND IN AN N -COMPONENT FERMI LIQUID IN THE RELAXATION TIME APPROXIMATION

We consider N independently conserved fermions, characterized in equilibrium by chemical potential μ_i and quasiparticle distribution functions

$$f_i^0(\mathbf{p}) = \Theta(\varepsilon_i^0(\mathbf{p}) - \mu_i) \quad (2)$$

where Θ is the Fermi function, which it will suffice for our purposes to take

as a step function. In nonequilibrium configurations, to leading order in the deviation from uniform equilibrium,

$$\delta n_i(\mathbf{p}) = n_i(\mathbf{p}) - f_i^0(\mathbf{p}) \quad (3)$$

the quasiparticle energies acquire the form

$$\begin{aligned} \varepsilon_i(\mathbf{p}) &= \varepsilon_i^0(\mathbf{p}) + \delta \varepsilon_i(\mathbf{p}) \\ &= \varepsilon_i^0(\mathbf{p}) + \sum_j \int f_{ij}(\mathbf{p}, \mathbf{p}') \delta n_j(\mathbf{p}') \frac{d^3 p'}{(2\pi)^3} \end{aligned} \quad (4)$$

The simplest (correct) form of the relaxation time approximation expresses the fact that collisions drive the system toward local equilibrium, by taking $I(n_i)$ to be of the form

$$I(n_i) = -(1/\tau)[n_i(\mathbf{p}) - \Theta(\varepsilon_i^0(\mathbf{p}) + \delta \varepsilon_i(\mathbf{p}) - \mu_i - \mathbf{p} \cdot \mathbf{u} - \delta \mu_i)] \quad (5)$$

The incorrect form (1) results from omitting the term $\delta \varepsilon_i(\mathbf{p})$, thereby overlooking the fact that the quasiparticle velocity-momentum relations in local equilibrium depend on the local form of the nonequilibrium distribution. At various points below we shall mention the consequences of such an omission.

To linear order in the deviations from equilibrium Eq. (5) assumes the form

$$I(n_i) = \frac{1}{\tau} \left(\frac{\partial f_i^0}{\partial \varepsilon_i^0} \right) \left[\sum_j (1 + F)_{ij} v_j - \mathbf{p} \cdot \mathbf{u} - \delta \mu_i \right] \quad (6)$$

where

$$\delta n_i(\mathbf{p}) = v_i(\hat{\mathbf{p}}) (-\partial f_i^0 / \partial \varepsilon_i^0) \quad (7)$$

and we define the dimensionless Fermi liquid parameters by

$$f_{ij}(\hat{\mathbf{p}}, \hat{\mathbf{p}}') g_j = F_{ij}(\hat{\mathbf{p}}, \hat{\mathbf{p}}') = \sum_{l=0}^{\infty} F_{ij}^l P_l(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}') \quad (8)$$

where $P_l(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')$ is the Legendre polynomial of order l . Here g_j is the density of states for the j th component,

$$g_j = m_j^* p_j / 2\pi^2 \quad (9)$$

and p_j and m_j^* are the Fermi momentum and effective mass for the j th component. We use an operator notation in which

$$(F_{ij} v_j)(\hat{\mathbf{p}}) = \int F_{ij}(\hat{\mathbf{p}}, \hat{\mathbf{p}}') v_j(\hat{\mathbf{p}}') \frac{d\Omega_{\mathbf{p}'}}{4\pi} \quad (10)$$

The functions $\delta \mu_i(\mathbf{r}, t)$ and $\mathbf{u}(\mathbf{r}, t)$ are determined by requiring that collisions conserve the density of each separate component and the total

momentum:

$$\begin{aligned} \int I(n_i) d^3 p / (2\pi)^3 &= 0, \quad i = 1, \dots, N \\ \sum_i \int I(n_i) \mathbf{p} d^3 p / (2\pi)^3 &= 0 \end{aligned} \quad (11)$$

[The condition of energy conservation follows from those of number conservation since $I(n_i(p))$ is proportional to $-\partial f_i^0 / \partial \varepsilon_i^0 = \delta(\varepsilon_i^0 - \mu_i)$; thus the linearized theory is consistent with energy conservation whether or not the term in $\delta \varepsilon_i$ is included in Eq. (5).] To linear order these $N+1$ conditions give

$$\begin{aligned} \delta \mu_i &= \sum_{j=1}^N \langle (1 + F)_{ij} v_j \rangle, \quad i = 1, \dots, N \\ \mathbf{u} &= \sum_{j,k=1}^N 3 g_j p_j \langle \hat{\mathbf{p}} (1 + F)_{jk} v_k \rangle / \sum_i g_i p_i^2 \end{aligned} \quad (12)$$

so that

$$I(n_i) = \frac{1}{\tau} \left(\frac{\partial f_i^0}{\partial \varepsilon_i^0} \right) \left[\bar{v}_i - \langle \bar{v}_i \rangle - \sum_j 3 A_{ij} \hat{\mathbf{p}} \cdot \langle \hat{\mathbf{p}} \bar{v}_j \rangle \right] \quad (13)$$

where

$$\bar{v}_i = \sum_j (1 + F)_{ij} v_j \quad (14)$$

$$A_{ij} = a_i b_j, \quad a_i = p_i / \sum_i g_i p_i^2, \quad b_j = g_j p_j \quad (15)$$

Had the term in $\delta \varepsilon_i$ been omitted from Eq. (5), the structure of Eq. (13) would have been the same except that \bar{v}_i would have been replaced by v_i . In the one-component case this is precisely Eq. (1). When there is just one component this modification has no effect on the structure of the zero-eigenvalue eigenfunctions of the collision term, which is why the damage only shows up in quantities that depend on the relaxation time τ . When there are many components, however, because F^1 is a matrix, the structure of the zero-eigenvalue eigenfunctions of the collision term does depend on whether \bar{v}_i or v_i appears in Eq. (13), and the failure to recognize that it should be \bar{v}_i leads to incorrect results even for quantities that do not depend on τ . We illustrate this in the case of the velocity of hydrodynamic sound.

The Fourier-transformed kinetic equation has the form

$$-i\omega v_i + i\mathbf{q} \cdot \mathbf{v}_i \bar{v}_i = -\frac{1}{\tau} \left[\bar{v}_i - \langle \bar{v}_i \rangle - \sum_j 3 A_{ij} \hat{\mathbf{p}} \cdot \langle \hat{\mathbf{p}} \bar{v}_j \rangle \right] \quad (16)$$

The solution symmetric about the direction of \mathbf{q} is given by $\nu = (1 + F)^{-1} \bar{\nu}$, where

$$\bar{\nu} = [1 - i\omega\tau(1 + F)^{-1} + iqV \cos \theta \tau]^{-1} (\langle \bar{\nu} \rangle + 3A \cos \theta \langle \bar{\nu} \cos \theta \rangle) \quad (17)$$

Here V is a diagonal matrix whose entries are the Fermi velocities, $\bar{\nu}$ is an N -component vector, and θ is the angle between $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$. Since qV/ω is of order unity, expanding the inverse operator to leading order in $\omega\tau$ gives

$$\bar{\nu} = [1 + i\omega\tau(1 + F)^{-1} - iqV \cos \theta \tau] (\langle \bar{\nu} \rangle + 3A \cos \theta \langle \bar{\nu} \cos \theta \rangle) \quad (18)$$

Equation (18) requires the consistency conditions

$$(1 + F^0)^{-1} \langle \bar{\nu} \rangle - \frac{qV}{\omega} A \langle \bar{\nu} \cos \theta \rangle = 0 \quad (19)$$

$$\frac{iqv\tau}{3} \langle \bar{\nu} \rangle + \left[1 - \left(1 + i\omega\tau \left(1 + \frac{F^1}{3} \right)^{-1} \right) A \right] \langle \bar{\nu} \cos \theta \rangle = 0 \quad (20)$$

Using the first of these to eliminate $\langle \bar{\nu} \rangle$ from the second, we have

$$\langle \bar{\nu} \cos \theta \rangle_i = \sum_{j,k} O_{ij} A_{jk} \langle \bar{\nu} \cos \theta \rangle_k \quad (21)$$

where

$$O_{ij} = \left[1 + i\omega\tau \left(1 + \frac{F^1}{3} \right)^{-1} - \frac{iq^2 V \tau}{3\omega} (1 + F^0 V) \right]_{ij} \quad (22)$$

Since A_{ij} has the simple factored form $a_i b_j$, Eq. (21) has a solution provided

$$\sum_{ij} b_i O_{ij} a_j = 1 \quad (23)$$

This gives for the sound velocity $c = \omega/q$,^{*†}

$$c^2 = \frac{\sum_{i,j} \frac{1}{3} p_i g_i v_i (1 + F^0)_{ij} p_j v_j}{\sum_{i,j} p_i g_i (1 + F^1/3)_{ij}^{-1} p_j} \quad (24)$$

If one follows the same procedure with $\bar{\nu}$ replaced (incorrectly) by ν on the right side of Eq. (16), one finds the incorrect expression for c^2 ,

$$\frac{\sum_{i,j} \frac{1}{3} p_i g_i [V(1 + F^0)V(1 + F^1/3)]_{ij} p_j}{\sum_i g_i p_i^2} \quad (25)$$

^{*}In the two-component case this expression differs from that found in Czerwonko's analysis⁵ of the kinetic equation for a spin-polarized Fermi liquid. His error can be traced to his imposition of a condition of constant spin density s , which is incompatible with the actual dynamics (as is evident if one considers a completely polarized system). Unlike his result, the correct sound velocity (24) is guaranteed to be real by the stability conditions he quotes. The correct sound velocity for the spin-polarized Fermi liquid, essentially in the form of our Eq. (29) below, is given by Meyerovich.⁶

[†]The derivation of this result has not required truncating the expansion of $F_{ij}(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')$ in Legendre polynomials.

As remarked above, the disagreement arises from the fact that F^1 is a matrix, so that the two results agree in the one-component limit. They also agree when p_i , g_i , v_i , and $\sum_j F_{ij}^1$ are all independent of i (as in an unpolarized liquid of spin- s fermions, $N = 2s + 1$).

We conclude by showing that the sound velocity c in Eq. (24) is indeed the correct hydrodynamic velocity,

$$c^2 = \left(\frac{\partial P}{\partial \rho} \right)_{n_i/n} \quad (26)$$

where P is the pressure, n is the total number density $n = \sum_i n_i$,

$$n_i = \int n_i(\mathbf{p}) \frac{d^3 p}{(2\pi)^3} \quad (27)$$

and $\rho = \sum_i m_i n_i$ is the total mass density, m_i being the bare particle mass of the i th component. The zero temperature Gibbs-Duhem relation is

$$dP = \sum_i n_i d\mu_i \quad (28)$$

so that

$$c^2 = \frac{1}{\rho} \sum_i n_i n_j \left(\frac{\partial \mu_i}{\partial n_j} \right)_{n_k} \quad (29)$$

The derivative $\partial \mu_i / \partial n_j$ is given, as in the one-component case, by noting that in equilibrium, changes in the chemical potentials are accompanied by changes in the number densities, which are related by

$$n_i = g_i \left(\delta \mu_i - \sum_j f_{ij}^0 \delta n_j \right) \quad (30)$$

Then Eq. (29) becomes

$$c^2 = \frac{1}{\rho} \sum_{ij} n_i (1 + F^0)_{ij} \frac{n_j}{g_j} \quad (31)$$

Since $n_i = p_i g_i v_i / 3$, this agrees with Eq. (24) provided the sum in the denominator of Eq. (24) is just 3ρ . This last equality follows from the N -component effective mass sum rules:

In arbitrary nonequilibrium configurations the momentum density must equal the total mass current,

$$\sum_i \int n_i(\mathbf{p}) \mathbf{p} \frac{d^3 p}{(2\pi)^3} = \sum_i \int m_i n_i(\mathbf{p}) \frac{\partial \epsilon_i(\mathbf{p})}{\partial \mathbf{p}} \frac{d^3 p}{(2\pi)^3} \quad (32)$$

As in the one-component case, taking the linear variation δ of both sides

of Eq. (32) gives*

$$g_i p_i = g_i p_i \frac{m_i}{m_i^*} + \sum_j g_j p_j \frac{m_j}{m_j^*} \frac{F_{ji}^1}{3} \quad (33)$$

from which it follows directly that

$$\sum_{ij} p_i g_i \left(1 + \frac{F_{ji}^1}{3}\right)^{-1} p_j = \sum_j \frac{m_j}{m_j^*} p_j^2 g_j = 3\rho \quad (34)$$

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*When specializing to the case $N=4$, these relations agree with those Oliva and Ashcroft⁷ derive for a Fermi liquid of two spin-1/2 components. In the case $N=2$ they agree with the relation for spin-polarized ³He given by Bashkin and Meyerovich⁸ (Eq. (4.2.5)), but not with the relation quoted more recently by Meyerovich,⁶ unless $B_{\pm}^{(1)}$ is multiplied by a factor p_{\mp}/p_{\pm} in his Eq. (A.5).